The statement of Theorem 10.10 on page 276 includes the premise \( k \geq 3 \). That premise is unnecessarily restrictive.

1. Give an example showing that the theorem’s conclusion holds also for \( k = 2 \).

2. Can the premise be loosened to \( k \geq 1 \) or only to \( k \geq 2 \)? Either give an example with \( k = 1 \) or explain why none exists.

Note that the book’s proof treats \( k = 3 \) and \( k = 4 \) as both being base cases and uses the inductive case only for \( k \geq 5 \). This isn’t necessary; you should be able to convince yourself that the inductive case would work fine for \( k = 4 \), allowing the base case to be limited to \( k = 3 \). The next question is whether this same process can be extended even further.

3. Start with the smallest example graph you found in the parts 1 and 2 of this homework. Label that graph with a minimum coloring. Apply to this graph the construction described in the proof’s inductive case so as to produce a larger graph. Label the vertices of this larger graph with a coloring produced as described in the portion of the proof starting “We can extend . . . .”

4. Is the coloring of the larger graph a minimum coloring?

5. Is the larger graph triangle free?

6. How would you edit the book’s proof to make it cover as broad a range of \( k \) values as possible and use as few base cases as possible?