Some Proofs about Distances and Centers

MCS-236

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**Theorem 1** If $G$ is a graph with radius $\text{rad} G$ and diameter $\text{diam} G$, then $\text{rad} G \leq \text{diam} G \leq 2 \text{rad} G$.

**Proof.** Because the radius is the minimum eccentricity of any vertex and the diameter is the maximum, the radius cannot be larger than the diameter. Let $u$ and $v$ be vertices of $G$ such that $d(u, v) = \text{diam} G$. Let $w$ be a central vertex so that $e(w) = \text{rad} G$. This means that no vertex is at a distance greater than $\text{rad} G$ from $w$. In particular $d(u, w)$ and $d(v, w)$ are both less than or equal to $\text{rad} G$. Therefore, $d(u, w) + d(v, w) \leq 2 \text{rad} G$. By the triangle inequality, $d(u, v) \leq d(u, w) + d(v, w)$. This establishes that $\text{rad} G \leq \text{diam} G \leq 2 \text{rad} G$.

**Theorem 2** For any graph $G$, there is some graph $H$ that has $G$ as its center.

**Proof.** We can construct $H$ by adding four vertices to $G$: $i_1, i_2, o_1, \text{ and } o_2$. The new edges are $e_1i_1, o_2i_2$, and for all $v$ in $V(G)$, $vi_1$ and $vi_2$. The eccentricity within $H$ of all vertices in $V(G)$ is 2, whereas the eccentricity of the added vertices is 3 for the $i$ vertices and 4 for the $o$ vertices.

**Theorem 3** For a graph $G$, there exists a graph $H$ that has $G$ as its periphery if and only if all vertices in $G$ have eccentricity 1 or no vertices in $G$ have eccentricity 1.

**Proof.** If all vertices in $G$ have eccentricity 1, then $G$ can itself serve as $H$. On the other hand, if no vertices in $G$ have eccentricity 1, then $H$ can be formed by adding one new vertex, $s$, and for each vertex $v$ in $V(G)$, the edge $sv$.

To show the converse, suppose that $G$ has a vertex $u$ that has eccentricity 1, other vertices $v$ and $w$ that have eccentricities greater than 1, and yet $G$ is the periphery of some graph $H$. We show this leads to a contradiction.
We know that the diameter of $G$ is greater than 1. Because $G$ is an induced subgraph of $H$, the diameter of $H$ is also greater than 1. Since $G$ is the periphery of $H$, any vertex in $V(G)$, such as $u$, must have $e_H(u) = \text{diam } H > 1$. Since $e_G(u) = 1$, there must be some vertex $s$ in $V(H) - V(G)$ that $u$ is farthest from. However, $s$ also has eccentricity equal to $e_H(u)$ yet is not included in the periphery, producing a contradiction.

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