Proposal for Sabbatical Leave

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2. At the time of the leave, I will have completed six years of full-time service to the college.

3. I am requesting a full year leave for the 2004-2005 academic year.

4. I have not previously received a compensated (regular sabbatical) leave from Gustavus. I did receive a “fourth year course reduction” in the spring of 2002.

   The most tangible accomplishments of my fourth year course reduction are as follows:
   
   (a) finished, submitted and had accepted by the *Journal of Algebra* a paper on test ideals.
   
   (b) gave an invited talk in a special session on commutative algebra at a sectional meeting of the American Mathematical Society (AMS).
   
   (c) applied for and received a travel grant from the Association for Women in Mathematics (AWM) which allowed me to attend three conferences in Italy and speak at one during the summer.
   
   (d) with two colleagues, developed and launched a new website for commutative algebra (www.commalg.org).
Overview

During the proposed leave, I will be conducting original research in mathematics. My research will focus on several related problems in commutative algebra. I also plan to broaden my area of research in an attempt to find research problems appropriate for undergraduates.

Background and previous work

My research so far has been in the area of commutative algebra, specifically commutative ring theory. Rings are abstract algebraic objects. One can think of a ring as a generalization of the integers. Some familiar examples of rings are the integers, the rational numbers, the real numbers and the complex numbers. Another familiar ring is the set of polynomials. I often study quotients rings of polynomial rings. In a quotient ring, two polynomials are considered equivalent if they differ by a multiple of one fixed polynomial. For example, in the ring $\mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3)$, $zx^3 + zy^3$ is equivalent to $-z^4$ since $zx^3 + zy^3 - (-z^4) = z(x^3 + y^3 + z^3)$ is a multiple of $x^3 + y^3 + z^3$. An ideal in a ring is a subring which is closed under multiplication by ring elements. For example, the even integers form an ideal of the integers. If you multiply an even integer by any integer, even or odd, you still have an even integer. One way to study the structure of a ring is to study its ideals.

My dissertation and most of my subsequent work focused primarily on tight closure, a prime characteristic or characteristic $p$ method. The idea of the tight closure of an ideal was introduced by Mel Hochster and Craig Huneke in the late 1980’s; it is a powerful tool that has been used to provide short proofs of some difficult theorems as well as new and unexpected theorems. Despite the apparent power of tight closure techniques, tight closure itself is very difficult to compute. One of the remaining open questions in tight closure theory is the localization problem. My dissertation dealt with several questions related to the localization problem. I showed that a certain mapping property implies that tight closure commutes with localization and that a certain class of rings has this mapping property. I also showed that tight closure is the same as two other closure operations, Frobenius closure and plus closure, in a certain class of ideals in cubical cone rings. I published three papers in refereed journals on these topics and gave several invited talks at AMS conferences.

My next project focused on test ideals. Test ideals play an important role in tight closure. Often the only way to actually compute tight closure is to determine and use the test ideal. Much is known about the existence of test ideals, but there are very few cases where we can actually compute the test ideal. I received an RSC grant to work on computing test ideals. I also spent much of my fourth year course reduction working on problems involving test ideals. I have published two papers and given several invited talks based on this research.

Tight closure techniques exploit the Frobenius endomorphism in characteristic $p$ and have proved to be very powerful tools in commutative algebra. My proposed research will make use of the Frobenius endomorphism and characteristic $p$ methods but will not address tight closure directly.
Description of Proposed Projects

Research in mathematics is often unpredictable, so it is odd to be proposing to work on specific problems two years from now. I can, however, describe the general area of research in which I plan to work, and I do have two specific problems in mind. My proposed research will address two general questions in commutative algebra.

The first question is related to classic results of Kunz. Kunz gave a characterization of a very well behaved class of rings (regular local rings) in positive characteristic when he showed that the length of certain quotients was bounded below and that regular local rings are characterized by achieving that lower bound. Rings for which that bound is small but is greater than the minimum should be “close” to regular local rings. In the case where the quotient of a regular local ring would have length $q^d$, I am interested in whether there are rings that have quotients of length $q^d + 1$ or other relatively small values, or whether there is a lower bound on the length when the ring is not regular. If such rings exist, I would like to characterize them. I have reason to believe that all such rings with quotients of small length may be F-regular or F-rational. F-regularity and F-rationality are two properties of rings that arise in the study of tight closure, and both are close to regularity.

My proposed research would build on preliminary results I obtained for lower bounds on the length of $R/m^p$. When estimating the length of $R/m^p$ or $R/m^{pe}$, one method involves estimating the minimal number of generators of a power of the maximal ideal ($m^{p+1}$ or $m^{pe+1}$). There is much work being done on finding upper bounds for the minimal number of generators of powers of the maximal ideal, however there is not nearly as much work being done to find lower bounds which is what is needed for the proposed research.

The second problem of my proposed research is to investigate an analogue of Macaulay’s theorem for Hilbert-Kunz functions. The Hilbert-Kunz function is a prime characteristic analogue of the Hilbert-Samuel function. The Hilbert-Kunz function gives, among other things, information about the complexity of the Frobenius powers of ideals. In the case of the Hilbert-Samuel function, Macaulay’s theorem gives upper bounds for the coefficients of the Hilbert-Samuel function in terms of the previous coefficients, in other words, bounds on the growth of the coefficients. It is also possible to construct ideals that achieve the maximum for Macaulay’s bound. Macaulay’s theorem is a very powerful tool for determining whether a given numerical function could be the Hilbert-Samuel function of some ring.

Results about Hilbert-Samuel functions have been extended to include prime characteristic, but there has not been much work done on the Hilbert-Kunz function, in part because so much less is known about it than about the Hilbert-Samuel function. The problem of finding an analogue of Macaulay’s theorem for the Hilbert-Kunz function was suggested to me by Huneke, and so, despite the apparent difficulty of anything related to the Hilbert-Kunz function, I am interested in working on this problem. Also, there has been some recent work on extending Macaulay’s theorem to other cases, in particular to the case of multigraded polynomial rings.

I would also like to expand my field of specialization by using Gröbner basis theory to study commutative rings. In particular, I propose to attempt to use computational algebra
techniques to study Hilbert-Kunz functions. It is my belief that expanding my specialization to include more computational techniques will allow me to make greater connections between my scholarship and my teaching and to increase the likelihood of finding projects suitable for research with students.

I plan to take several short trips to visit the University of Kansas and work with Huneke during my leave. The leave will also give me the flexibility to attend many conferences and workshops. It is possible that I may want to visit KU or somewhere else for a more extended period of time. I do not anticipate any difficulties with this since there is no money involved and I don’t need anything other than access to mathematicians and occasionally a library. I was a Visiting Researcher at KU during my fourth year course reduction and Huneke has confirmed that I could visit the department again. Visiting other departments would be helpful but is not essential to carrying out my proposed research.

Preparation Leading up to the Sabbatical

During my fourth year course reduction, I visited the University of Kansas and worked with Craig Huneke. At that time I started working on the first proposed problem and obtained some preliminary results. During my stay Huneke suggested the second proposed problem to me. I applied for an AWM Mentoring Travel Grant to spend a month working with Huneke in June 2003. If my application is successful, I will have money left over that could be used to fund another month long visit to Kansas during 2004-2005.

I attended three workshops at the Mathematical Sciences Research Institute this year that were part of a year long program in commutative algebra. One of these workshops was about computational commutative algebra. I attend, on average, three conferences a year, and I will continue to do that next year in order to stay up to date on current research in commutative algebra.

Relation of Proposed Projects to Previous Work

The research I plan to conduct during my sabbatical leave will allow me to build upon an already existing program of research using prime characteristic methods in commutative algebra.

Expected Outcomes and Future Activities

I expect that the proposed work will result in a paper which I will submit to an appropriate journal. I also expect to make presentations at national and regional conferences. In addition, I will be in a better position to write an NSF proposal. If I am successful at generating problems appropriate for undergraduates, I will also be better able to write a grant proposal for the NSF’s RUI (Research at Undergraduate Institutions) program.
Value of projects

My plan for scholarly activity in the next several years includes two broad goals. The first is to be able to keep up with a fast moving research area and to continue to be able to carry out and publish research on commutative algebra in quality journals. When I attend conferences, I often find that I am the only participant from a small liberal arts college. It is challenging to conduct research in an area in which most of the players have significantly more time to devote to their research. A full year leave would help me tremendously to maintain a viable research program.

My second goal is related to research with undergraduates. Tight closure is not a very accessible topic for undergraduates. Even commutative algebra in general is not a topic that is covered by the standard undergraduate curriculum. It is my hope that continuing to learn more about open problems in computational algebra will allow me to find interesting and accessible problems for student-faculty research.

Plans for Public Presentation of Leave Results

I expect to present the results of my research in a departmental seminar at Gustavus. In addition, I expect to present my research results to a relatively specialized audience at a professional meeting. It is unlikely that my research would be appropriate for a more general audience, but I have given a Shop Talk before, and I will give one again, if I can think of a way to make it of interest to a very general audience.