HW 2.26

The Class

MCS-236, Fall 2011

Two graphs, G and $\overline{G}$, are complementary if they have the same vertex sets but have opposite edge sets in the following sense. For any two vertices s and v, the edge sv is either in $E(G)$ or in $E(\overline{G})$, but not both.

If G and $\overline{G}$ are complementary graphs of order n, we can clarify their relationship by reference to the complete graph K_n. The edge sets $E(G)$ and $E(\overline{G})$ partition $E(K_n)$.

The degree of a vertex v in a graph G is the number of edges incident with v, which we will denote as $\deg_G v$. Similarly, we will denote the degree of v in the graph $\overline{G}$, as $\deg_{\overline{G}} v$.

Although we will turn our attention to regular graphs, we can first prove a more general result that applies to any complementary pair of graphs.

Lemma 1 *For any vertex v in a graph G of order n,* $\deg_G v + \deg_{\overline{G}} v = n-1$.

Proof. Since G and $\overline{G}$ are complementary their edge sets partition the complete graph K_n. In K_n, each vertex is incident to $n-1$ edges. Therefore $\deg_G v + \deg_{\overline{G}} v = n-1$. ∎

Theorem 1 *A graph G is regular if and only if $\overline{G}$ is regular.*

Proof. The graph G is regular, so each vertex in G has the same degree, r. By Lemma 1, $r + \deg_{\overline{G}} v = n-1$. Therefore, each vertex in $\overline{G}$ has degree $n-1-r$. Thus $\overline{G}$ is regular. ∎

Theorem 2 *If G and $\overline{G}$, two complementary graphs of order n, are both r-regular for some integer r, then n is odd.*

Proof. Because G and $\overline{G}$ are both r-regular, the degree of any vertex is r in each graph. By Lemma 1, $r+r = n-1$. Solving for n, we have $n = 2r+1$. Since r is an integer, then $2r+1$ is an odd integer, so n is odd. ∎